GOODWILLIE'S CALCULUS VIA RELATIVE HOMOLOGICAL ALGEBRA. THE ABELIAN CASE

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1. Introduction

We will explain how elementary concepts of relative homological algebra yield the Taylor tower for functors from pointed categories to abelian groups recovering the constructions of Johnson and McCarthy [2],[3].

Let $\mathbf{C}$, $\mathbf{D}$ be abelian categories with enough projective objects. Let $i_*:\mathbf{C}\to\mathbf{D}$ and $i^*:\mathbf{D}\to\mathbf{C}$ be functors, such that i^* is left adjoint to i_*. We will assume that i_* is full and faithful and exact. After taking the left derived functors one obtains a pair of adjoint functors $(\mathsf{L}(i^*)\vdash \mathsf{L}(i_*))$ between the derived categories $\mathsf{D}^-(\mathbf{D})$ and $\mathsf{D}^-(\mathbf{C})$. In general, $\mathsf{L}(i_*):\mathsf{D}^-(\mathbf{C})\to\mathsf{D}^-(\mathbf{D})$ is not a full embedding. Instead one defines a full subcategory $\mathsf{D}^-_{\mathbf{C}}(\mathbf{D})$ of $\mathsf{D}^-(\mathbf{D})$ by

$$\mathsf{D}^-_{\mathbf{C}}(\mathbf{D})=\{X_*\in\mathsf{D}^-(\mathbf{D})\,|\,H_n(X_*)\in\mathbf{C},n\in\mathbb{Z}\}.$$

Denote by $j_*:\mathsf{D}^-_{\mathbf{C}}(\mathbf{D})\to\mathsf{D}^-(\mathbf{D})$ the full inclusion. Then the functor $\mathsf{L}(i_*)$ factors through j_*. In the favourable cases the functor j_* has left adjoint j^*, however we do not know whether j^* always exists. In the next section we will construct the functor j^* under certain circumstances. Our construction is based on the elementary results of the relative homological algebra [1] and is probably well-known. In the last section we explain how the results of Section 2 imply the main results of [2],[3].

In [4] we will extend our method from abelian to nonabelian case.

2. The main construction

Let $\mathcal{A}$ be an abelian category with coproducts and let $\mathcal{P}$ be a set of objects in $\mathcal{A}$ such that each $P\in\mathcal{P}$ is projective. Define the following full subcategory

$$\mathcal{B}=\mathcal{P}^{\perp}=\{A\in\mathcal{A}\,|\,Hom_{\mathcal{A}}(P,A)=0,P\in\mathcal{P}\}.$$

It is clear that $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$. That is, $\mathcal{B}$ is closed under taking kernels, cokernels and extensions. In particular, $\mathcal{B}$ is also abelian. Denote by $i_*:\mathcal{B}\to\mathcal{A}$ the inclusion. Then i_* is exact.

For any $A\in\mathcal{A}$ one puts

$$\Phi(A)=\bigoplus_{f:P\to A}P.$$

Here P runs through all objects of $\mathcal{P}$. For a morphism $f:P\to A$ we let $in_f:P\to\Phi(A)$ be the standard inclusion. Define $\epsilon_A:\Phi(A)\to A$ by $\epsilon_A\circ in_f=f$ and denote $\mathsf{Coker}(\epsilon_A)$ by $i^*(A)$. Since $\mathsf{Hom}_{\mathcal{A}}(P,\epsilon_A)$ is surjective one sees that $i^*(A)\in\mathcal{B}$. In this way one obtains a functor $i^*:\mathcal{A}\to\mathcal{B}$ which is left adjoint to i_*.

A morphism $f:X\to Y$ in $\mathcal{A}$ is called $\mathcal{P}$-epimorphism provided $\mathsf{Hom}_{\mathcal{A}}(P,f):\mathsf{Hom}_{\mathcal{A}}(P,X)\to\mathsf{Hom}_{\mathcal{A}}(P,Y)$ is surjective. For example, for any object $A\in\mathcal{A}$ the morphism $\epsilon_A:\Phi(A)\to A$ is a $\mathcal{P}$-epimorphism. Hence $\mathcal{P}$ is a projective class in the sense of [1] and therefore by [1, Proposition 3.1] any object A has a $\mathcal{P}$-projective resolution. Thus there is a chain complex (X_*,d) such that $X_n=0$ if $n<-1$, $X_{-1}=A$, $X_n\in\mathcal{P}$ for any $n\geq 0$ and for any $P\in\mathcal{P}$ the following sequence is exact:

$$\cdots\to\mathsf{Hom}_{\mathcal{A}}(P,X_n)\to\cdots\to\mathsf{Hom}_{\mathcal{A}}(P,X_0)\to\mathsf{Hom}_{\mathcal{A}}(P,X_{-1})\to 0.$$

It follows that $X_* \in \mathsf{D}^-_{\mathcal{B}}(\mathcal{A})$. By the standard properties of $\mathcal{P}$-projective resolutions the assignment $A \mapsto X$ extends to a functor $j^* : \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}^-_{\mathcal{B}}(\mathcal{A})$ which turns to be left adjoint to j_*.

Assume now that instead of a single set $\mathcal{P}$, a descending sequence of sets

$$\cdots \subset \mathcal{P}_n \subset \mathcal{P}_{n-1} \subset \cdots \subset \mathcal{P}_1$$

is given, each of which satisfies the assumptions made in the beginning of Section 2. One obtains abelian categories $\mathcal{B}_n = \mathcal{P}_n^\perp$ and functors $i_{n*}, i_n^*, j_{n*}, j_n^*$. Clearly, $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{B}_3 \subset \cdots \subset \mathcal{A}$ and for any object $A \in \mathcal{A}$ one obtains the towers of epimorphisms

$$A \to \cdots \to i_* i_n^*(A) \to i_* i_{n-1}^*(A) \to \cdots \to i_* i_2^*(A) \to i_* i_1^*(A)$$

and of morphisms in $\mathsf{D}^-(\mathcal{A})$

$$A \to \cdots \to j_* j_n^*(A) \to j_* j_{n-1}^*(A) \to \cdots \to j_* j_2^*(A) \to j_* j_1^*(A).$$

3. Applications to Goodwillie's calculus

Let $\mathbf{M}$ be a small category with zero object 0 and finite coproduct $\vee$. We let $\mathcal{A}$ be the category of all functors from $\mathbf{M}$ to the category of abelian groups. Then $\mathcal{A}$ is an abelian category with enough projective objects. The functors h_a are small projective generators of $\mathcal{A}$. Here a is running through all objects of the category $\mathbf{M}$ and $h_a \in \mathcal{A}$ is given by $h_a = \mathbb{Z}[\mathsf{Hom}_{\mathbf{M}}(a, -)]$. The obvious maps $a \to 0 \to a$ yield a splitting $h_a = \bar{h}_a \oplus \mathbb{Z}$, where $\mathbb{Z} = h_0$ is the constant functor with values equal to $\mathbb{Z}$. Thus the collections $\bar{h}_a$, $a \in \mathbf{M}$ together with $\mathbb{Z}$ also form a family of small projective generators. Clearly $h_{a\vee b} = h_a \otimes h_b$. It follows that the level-wise tensor product of projective objects is again a projective object. For any natural number $n \geq 1$ we let $\mathcal{P}_n$ be the collection of projective objects of the form $\bar{h}_{a_1} \otimes \cdots \otimes \bar{h}_{a_k}$, $k > n$. One easily checks that the corresponding category $B_n = \mathcal{P}_n^\perp$ is the category of functors of degree $\leq n$ (in the sense of Eilenberg-MacLane), while $\mathsf{D}^-_{\mathcal{B}_n}(\mathcal{A})$ is equivalent to the category of functors from $\mathbf{M}$ to the category of chain complexes of abelian groups of degree $\leq n$ (in the sense of Goodwillie). This follows from the fact that $\mathsf{Hom}_{\mathcal{A}}(\bar{h}_{a_1} \otimes \cdots \otimes \bar{h}_{a_k}), T) = cr_k T(a_1 \cdots, a_k)$, where cr_k is the k-th crossed-effect [3]. The last isomorphism is a trivial consequence of the Yoneda lemma and the decomposition rule: $h_{a\vee b} = h_a \otimes h_b$. It follows that in this situation the towers constructed in Section 2 and the ones constructed in [2],[3] are equivalent.

References

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